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A glueing process for rings with polynomial identity

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INTRODUCTION

The (commutative) notions of seminormalization arise in the work of Andreotti and Norguet [3] and Andreotti and Bombieri [2], where it was noticed that, roughly speaking, normalization is too coarse a technique to be applied to the study and classification of certain moduli problems. The main problem in this context is that "too many" points split up in the normalizing process. In order to remedy this, one has to glue together points which were split up in the act of normalizing. In the work of Traverso [15] and subsequently of Tamone [14] et al. a unified approach to these difficulties has been presented, where the notion of normalization has been replaced by the more natural notion of seminormalization of a ring (resp. a (noetherian) scheme). One of the main results in loc. cit. states that if a noetherian ring A is seminormal in an overring B which is a finite module over A, then A may be obtained from B by a finite number of "glueings over primes in A".

Now in studying the structure of PI rings from the geometric point of view (cf. [4, 5, 6, 10, 19, 21]) similar problems are encountered, i.e. one needs tools to get rid of singularities and one of them is a noncommutative counterpart of seminormalization. Full details will be worked out elsewhere. It appears that this construction too may be viewed as an iteration of glueings. The purpose of this note is to describe the above mentioned glueing process for PI rings. It will

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appear "en passant" that this procedure may also be applied to yield some interesting structural results on affine PI rings, in the vein of those described by Small and others.

1. GENERALITIES

(1.1.) A ringmorphism $\varphi : R \to S$ is called an *extension* (in the sense of Procesi [12]) whenever S is generated by $Z_R(S) = \{s \in S;\ \forall r \in R, \varphi(r)s = s\varphi(r)\}$ for the induced R-module structure. If S is generated by $Z(S)$, the center of S, we speak of a *central extension*. The main interest of studying extensions resides in the fact that any extension $\varphi : R \to S$ induces a continuous map

$$^{a}\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R) : P \to \varphi^{-1}(P)$$

for the Zariski topology. We call an extension $\varphi : R \to S$ *integral* if each $s \in S$ satisfies a relation of the form

$$s^n + r_{n-1}s^{n-1} + \ldots + r_0 = 0 \text{ or } s^n + s^{n-1}r_{n-1} + \ldots + r_0 = 0$$

for some $r_i \in \varphi(R)$.

(1.2.) We will not go into the details of noncommutative symmetric localization here, let us refer the reader to [8, 9, 16, 18] for full details and information. We will only recall the following. Assume R to be prime (only for simplicity's sake!). For each $P \in \mathrm{Spec}(R)$ we define $\mathscr{L}(R-P)$ to be the filter of left R-ideals generated by the ideals I of R with $I \not\subset P$. We then put

$$Q_{R-P}(R) = \varinjlim_{I \in \mathscr{L}(R-P)} \mathrm{Hom}(I, R),$$

where homomorphisms are to be left R-linear. If R possesses a classical ring of fractions $Q(R)$, e.g. if R satisfies a polynomial identity, due to Posner's theorem [12], or more generally, by putting $Q(R) = Q_{R-O}(R)$, it is easily verified that there is a canonical R-algebra isomorphism

$$Q_{R-P}(R) = \{q \in Q(R);\quad \exists I \in \mathscr{L}(R-P), Iq \subset R\}.$$

Anyway, even in the most general situation one may endow $Q_{R-P}(R)$ with a ringstructure extending that of R in a rather straightforward way. Moreover, one may easily check that there is a canonical inclusion $j_{R-P} : R \to Q_{R-P}(R)$ and that $Z(Q_{R-P}(R)) = Z_R(Q_{R-P}(R))$. One of the main properties of [21] may now be stated as:

(1.3.) PROPOSITION. Let R be a prime PI algebra and P a maximal ideal of R, then there exists a canonical surjection

$$\pi_{R,P} : Q_{R-P}(R) \to \mathbb{K}_R(P) = : Q(R/P) = R/P$$

with the property that

$$P = \mathrm{Ker}(\pi_{R,P} o j_{R-P}). \qquad \square$$

(1.4.) PROPOSITION. Let $\varphi : R \to S$ be an injective extension of prime *PI* rings. For each prime ideal Q of S and $P = \varphi^{-1}(Q)$ of R there exists a unique ringmorphism $\varphi_Q : Q_{R-P}(R) \to Q_{S-Q}(S)$ extending φ.

(1.5.) NOTE. If one wants to extend (1.3.) to arbitrary prime ideals or (1.4.) to arbitrary extensions, one has to define $Q^{bi}_{R-P}(R) = RZ_R(Q_{R-P}(R))$ endowed with the obvious map $j^{bi}_{R-P} : R \to Q^{bi}_{R-P}(R)$. By definition j^{bi}_{R-P} is then an extension, the localizing extension at P. One may then prove that if R is a left noetherian prime *PI* algebra and P is a prime ideal of R, then there exists a canonical central extension $\pi_{R,P} : Q^{bi}_{R-P}(R) \to R_R(P)$ such that $P = \mathrm{Ker}\,(\pi_{R,P} o j_{R-P})$. Moreover, if $\varphi : R \to S$ is an extension between left noetherian prime *PI* rings, then for each prime ideal Q of S and $P = \varphi^{-1}(Q)$ of R there is a unique ringmorphism $\varphi_Q : Q^{bi}_{R-P}(R) \to Q^{bi}_{S-Q}(S)$ extending φ; it automatically follows that φ_Q is an extension.

2. GLUEING POINTS

(2.1.) Let $i : R \to S$ be an injective morphism of prime *PI* rings. Let P be a maximal ideal of R and assume that there is a finite number $\{P_1, \ldots, P_n\}$ of maximal ideals of S lying over P, i.e. $P_\alpha \cap R = P$ for $1 \leq \alpha \leq n$. Note that we do not exclude the possibility of other prime ideals of S lying over P. We will primarily be interested in the following situations:

(2.1.1.) i is an integral extension; in this case $P_1, \ldots, P_n$ are the only *prime* ideals lying over P, due to *LO* and *GU*, cf. [13].

(2.1.2.) R is a commutative ring, e.g. a subring of the center of S; in this case any intermediate ring $R \subset T \subset S$ is still an extension of R;

(2.1.3.) i is a central extension, e.g. S is the central integral closure of R or the trace ring $T(R)R$, cf. [1]; in this case any intermediate ring $R \subset T \subset S$ yields a central extension $T \subset S$.

Denote by $\varepsilon_i : S \to S/P_i$ the canonical map, and let $\varepsilon = \oplus \varepsilon_i : S \to \oplus_i S/P_i$. Write $\Delta_i : \mathbb{K}_R(P) \to \mathbb{K}_S(P_i)$ for the canonical inclusion induced by P_i over the inclusion $R \to S$ and let $\Delta = \oplus \Delta_i : \mathbb{K}_R(P) \to \oplus\, \mathbb{K}_S(P_i)$. Let $D = \{s \in S;\ \varepsilon(s) \in \mathrm{Im}\,\Delta\}$. We will say that the ring D is *obtained from S by glueing over P*. Let us now describe some of its features. Clearly D maps into $\mathbb{K}_R(P)$ in the obvious way, we thus obtain a commutative diagram

$$\begin{tikzcd} D \arrow[r, "\mu"] \arrow[d] & \mathbb{K}_R(P) \arrow[d, "\Delta"] \\ S \arrow[r, "\varepsilon"'] & \oplus\, \mathbb{K}_S(P_i) \end{tikzcd}$$

Unless explicitly otherwise indicated all notations will be fixed throughout the rest of this paragraph. For related results in the commutative case, cf. [7, 14, 15].

(2.2.) PROPERTY $\mu(D) = \mathbb{K}_R(P)$ and $Q = \bigcap_i P_i$ is maximal in D.

PROOF. First let us note that the morphism $\varepsilon : S \to \bigoplus \mathbb{K}_S(P_i)$ is surjective with kernel $\cap P_i = Q$. Indeed, Ker $\varepsilon = Q$ is obvious, while the fact that the P_i are (co)maximal implies that ε is surjective, in view of the Chinese Remainder Theorem. Moreover $Q \subset D$, since for each $q \in Q$ we have $\varepsilon(q) = (0, \ldots, 0) = \Delta(0)$. Now, if $r \in \mathbb{K}_R(P)$, then we may find $b \in S$ such that $\varepsilon(b) = \Delta(r)$. But obviously $b \in D$, so μ is surjective. Finally, Ker $\mu = D \cap$ Ker $\varepsilon = D \cap Q = Q$, proving the maximality of Q in D. □

(2.3.) PROPERTY $P_1, \ldots, P_n$ are the only *prime* ideals of S lying over P.

PROOF. First note that $P_i \cap D = Q$ for each $1 \leq i \leq n$. Indeed, if $m \in P_i \cap D$, then $\varepsilon_i(m) = 0$. Hence for each $1 \leq j \leq n$ we have $\varepsilon_j(m) = 0$, by the choice of m. For, indeed $0 = \varepsilon_i(m) = \Delta_i \mu(m)$, hence $\mu(m) = 0$ as Δ_i is injective, so $\Delta_j \mu(m) = 0$ as well. It thus follows that $m \in \cap$ Ker $\varepsilon_j = Q$. Now the other inclusion is obvious, so indeed $P_i \cap D = Q$. If T is a prime ideal of S with the property that $T \cap R = Q$, then $P_1 \ldots P_n \subset \cap P_i = Q \subset T$, so for some i we have $P_i \subset T$ implying that $P_i = T$ since each P_i is maximal by assumption. □

(2.4.) PROPERTY. If $P' \in \operatorname{Spec}(S)$ and $P' \notin \{P_1, \ldots, P_n\}$ then $P' \cap D$ is a prime ideal of D and P' is the only prime ideal of S lying over it.

PROOF. First note that $D \subset S$ is not necessarily an extension, so $P' \cap D$ is not automatically a prime ideal of D. Yet, we know that $Q = \bigcap_i P_i$ is a common ideal of D and S, so, due to results by Amitsur and Small [1] we know that $P' \cap D$ is prime for each P' prime in S not containing Q. But $P' \not\supset Q$ is merely saying that $P' \notin \{P_1, \ldots, P_n\}$. Next, let P_1' be another prime ideal of S, lying over $P' \cap D$. We consider three possibilities:

(2.4.1.) $P_1' = P_i$ for some $1 \leq i \leq n$, then $P_1' \cap D = P_i \cap D = Q$, i.e.

$$P_1' \in \{P_1, \ldots, P_n\}$$

by the foregoing property; but then $P' \in \{P_1, \ldots, P_n\}$ as well, contradiction!

(2.4.2.) $P_1' \notin \{P_1, \ldots, P_n\}$ and $P' \subsetneq P_1'$, then $P' \cap D \neq P_1' \cap D$ by strong incomparability, cf. [1].

(2.4.3.) $P_1' \notin \{P_1, \ldots, P_n\}$ and $P' \not\subset P_1'$, then we may pick $r \in P' \cap Q, r \notin P_1'$ since otherwise $P' \cap Q \subset P_1$, implying $P' \subset P_1'$ or $Q \subset P_1'$, both being excluded; but then $P' \cap D \neq P_1' \cap D$, since $P' \cap Q \subset P' \cap D'$. □

(2.5.) COROLLARY D is a prime *PI* ring. □

(2.6.) PROPERTY. $Z(D) \subset Z(S)$ and equality if $Z(S) \subset R$.

PROOF. Since i is not necessarily an extension, this is not obvious. Now, if $z \in Z(D)$ and $s \in S$, then for each $q \in Q$ we have $q(zs) = (qz)s = (zq)s = z(qs) = (qs)z = q(sz)$, hence $Q(zs - sz) = 0$ in S, hence $zs - sz = 0$ as $Q \neq 0$ and S is prime. This proves that $z \in Z(S)$. Moreover, if $Z(S) \subset R$ we find $Z(S) \subset Z(D)$ as well. □

(2.7.) COROLLARY. $Q(D) = Q(S)$.

PROOF. If $q \in Q(D)$, then $q = dc^{-1}$ for some $d \in D$, $c \in Z(D) - \{0\}$, hence $q \in Q(S)$, since $D \subset S$, $Z(D) \subset Z(S)$. Conversely, if $q \in Q(S)$, then for some non-zero ideal I of S we have $Iq \subset S$. But then also $QIq \subset QS \subset D$ and $QI \neq 0$ is an ideal of D, i.e. $q \in Q(D)$. □

(2.8.) PROPERTY. If $P' \notin \{P_1, \ldots, P_n\}$ is a prime ideal in S, then $Q_{P' \cap D}(D) = Q_{P'}(S)$.

PROOF. Note that we have written $Q_{P'}$ instead of $Q_{S-P'}$, etc. First, choose $q \in Q_{P'}(S)$ then we may find $I \not\subset P'$ in S such that $Iq \subset S$. But then $QIq \subset QS \subset Q \subset D$, so if we check that QI (which is obviously an ideal of D') is not contained in $P' \cap D$, this will yield $q \in Q_{P' \cap D}(D)$. Now, if $QI \subset P' \cap D$, then $QI \subset P'$, i.e. $Q \subset P'$ or $I \subset P'$, both being excluded. Conversely, if $q \in Q_{P' \cap D}(D)$, then we may find $I \not\subset P' \cap D$, an ideal of D, such that $Iq \subset D$. But then $QIQq \subset S$ and $QIQ \not\subset P'$, since otherwise $Q \subset P'$ or $I \subset P$ (and hence $I \subset P' \cap D$), both being excluded. □

(2.9.) COROLLARY. If P' is a prime ideal in S not contained in $\{P_1, \ldots, P_n\}$, then $\mathbb{K}_D(P' \cap D) = \mathbb{K}_S(P')$.

PROOF. If $q \in Q(S/P')$, then for some $I \not\subset P'$ we find $Iq \subset S/P'$. Consider the following commutative diagram, where $P'' = D \cap P'$:

$$\begin{tikzcd} D/P'' \arrow[r, "\tilde{\alpha}"] \arrow[d, "\mathscr{J}_{P''}"'] & S/P' \arrow[d, "\mathscr{J}_{P'}"] \\ Q(D/P'') \arrow[r, "\alpha"'] & Q(S/P') \end{tikzcd}$$

Now, if $a \in S/P'$, then $Qa \subset \mathrm{Im}\ \tilde{\alpha}$, so in particular $QIq \subset \mathrm{Im}\ \alpha$, hence $q \in \mathrm{Im}\ \alpha$, since QI is an ideal of D with $QI \not\subset P''$. □

(2.10.) PROPERTY. Q is the only maximal ideal of D lying over P.

PROOF. Let $Q_1 \neq Q$ ly over P in D, then we may find Q_1' in S lying over Q_1; but then $Q_1' \cap R = P$, implying that $Q_1' \in \{P_1, \ldots, P_n\}$. Indeed, since Q_1 is assumed to be maximal unique lying over ($Q_1 \not\supset Q_1'$) implies that Q_1' is maximal as well. But then $Q_1 = Q_1' \cap D = P_\alpha \cap D = Q$. $\square$

(2.11.) NOTE. If $R \subset S$ is an integral extension, then Q is easily seen to be the only *prime* ideal of D lying over P.

3. AN APPLICATION

(3.1.) With notations as before, assume now that $R/P \to S/P_i$ is a finitely generated extension for each index i. This is the case for example if S is a finitely generated extension of R. Now S/P_i is finite dimensional over the center of $Q(R/P) = R/P$, cf. Procesi [11, 12]. It follows that S/P_i is a finitely generated (left) R/P-module say S/P_i is generated by $\{\bar{x}_1^{(i)}, \ldots, \bar{x}_{n_i}^{(i)}\}$, where $x_\alpha^{(i)} \in S$. Let us show that $S = D + \sum_{i,\alpha} D x_\alpha^{(i)}$.

Indeed, consider the commutative diagram:

$$\begin{tikzcd} D \arrow[r, "\mu"] \arrow[d] & \mathbb{K}_R(P) \arrow[d, "\Delta"] \\ S \arrow[r, "\varepsilon"'] & \oplus_i \mathbb{K}_S(P_i) \end{tikzcd}$$

Choose $s \in S$, then $\varepsilon_i(s) = \sum_\alpha \lambda_{\alpha,i} \bar{x}_\alpha^{(i)}$ for some $\lambda_{\alpha,i} \in k_R(P)$, say $\lambda_{\alpha,i} = \mu(d_{\alpha,i})$, with $d_{\alpha,i} \in D$. Then $\varepsilon(\sum_{\alpha,i} \lambda_{\alpha,i} x_\alpha^{(i)}) = 0$, i.e. $\sum_{\alpha,i} d_{\alpha,i} x_\alpha^{(i)} \in \operatorname{Ker} \varepsilon = Q \subset D$, proving the assertion.

Hence:

(3.2.) PROPOSITION. If for each i the canonical map $R/P \to S/P_i$ is a finitely generated extension, then S is a finitely generated D-module.

(3.3.) Now, let $T \neq P$ be another maximal ideal of R and assume that there is only a finite number of maximal ideals $\{T_1, \ldots, T_m\}$ of S lying over T. In passing from S to D we have seen that the $T_i \cap D$ stay maximal in D, by using "lying over" and "going up" outside of the common ideal Q. Moreover, the $T_i \cap D$ are the only maximal ideals of D lying over T. Indeed, if M is a maximal ideal of D with $M \cap R = T$, then $M \neq Q$, for otherwise $T = M \cap R = Q \cap R = P$, hence there exists a unique prime ideal M_1 lying over M. But then $M_1 \cap R = M_1 \cap D \cap R = M \cap R = T$, i.e. $M_1 \in \{T_1, \ldots, T_n\}$. Moreover, if S/T_i is a finitely generated extension of R/T for each index i, then so is $D/D \cap T_i$, since $\mathbb{K}_S(T) = \mathbb{K}_D(D \cap T_i)$. It is thus indeed clear that this glueing process may be used iteratively, on different maximal ideals.

(3.4.) EXAMPLE. Let R be finitely generated over a subring A of its center and assume that only a finite number of maximal ideals of A "split up" (in a finite number of maximal ideals), then we may apply the glueing process defined above on each maximal ideal successively. We then finally obtain an intermediate ring $A \subset D \subset R$ with the property that no splitting of maximal ideals occurs between A and D and such that R is a finite (left) D-module.

As an example if $A = k[x]$ and

$$R = \begin{pmatrix} k[x] & (x) \\ k[x] & k[x] \end{pmatrix},$$

then D consists of all

$$\begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{pmatrix} \in R \text{ with } f_{11}(0) = f_{22}(0).$$

More specifically, choose $A = Z(R)$. Now, if R is an affine prime PI algebra over its center and if R has Krull dimension one, then the above mentioned splitting requirement is automatically met, hence:

(3.5.) PROPOSITION. *If R is a prime PI ring of Krull dimension one, finitely generated over its center C, then R is a finite module over a Zariski central subring with center C.*

PROOF. Following Van Oystaeyen [17] a ring R with center C is a Zariski central ring if for each prime ideal P or R we have $P = \text{rad } R(P \cap C)$; for more details the reader is referred to loc. cit.

Now note that we just have to check that D is Zariski central. This follows from the fact that R has Krull dimension 1, i.e. it suffices to check that D has Krull dimension 1 too. Let us do this for the first step (notations as before). Pick a prime ideal T in D. If $T \neq Q$, then $T = T_1 \cap D$ for some prime, hence maximal ideal of R, hence T is maximal too, by "going up". □

More generally.

(3.6.) PROPOSITION. *Let R be a prime PI ring, finitely generated over its center C, assume that R has only a finite number of "bad" primes and assume that R is a Jacobson ring, then there exists a subring D of R with the property:*

(3.6.1.) *D is Zariski central with center C;*

(3.6.2.) *R is a finite D-module.*

PROOF. First, note that if $P \subset P'$ are prime ideals of R, then P "bad" implies P' "bad"; indeed, if $F(R)$ denotes the Formanek center of R, then $F(R) \subset P$ implies $F(R) \subset P'$. Since R is a Jacobson ring, it then follows that the only "bad" primes are maximal ideals. The conclusion now follows as before. □

NOTE. If R is integral over C, then finiteness over C follows.

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